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On the number of positive solutions of a nonlinear algebraic system

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Abstract

In this paper, we study the nonlinear algebraic system of the form

$$x = \lambda A F(x), \tag{E}$$

where $\lambda > 0$ is a parameter, x and $F(x)$ denote the column vectors

$$\operatorname{col}(x_1, x_2, \ldots, x_n) \quad \text{and} \quad \operatorname{col}(f_1(x_1), f_2(x_2), \ldots, f_n(x_n)),$$

respectively with $f_k : R \to R$, $k \in \{1, 2, \ldots, n\} = [1, n]$ and n is a positive integer. $A = (a_{ij})_{n\times n}$ is an $n \times n$ matrix and all its entries are positive numbers.

Many problems in various areas such as difference equations, boundary value problems, dynamical networks, stochastic process, numerical analysis etc. can be converted to system (E). Applying fixed point theorems, we prove results on existence, uniqueness, multiplicity and nonexistence of positive solutions for (E).

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1. Introduction

Consider the nonlinear algebraic system of the form

$$x = \lambda A F(x), \tag{1.1}$$

where $\lambda > 0$ is a parameter,

$$x = \mathrm{col}(x_1, x_2, \ldots, x_n) \quad \text{and} \quad F(x) = \mathrm{col}(f_1(x_1), f_2(x_2), \ldots, f_n(x_n))$$

are column vectors with $f_k : R \to R$ for $k \in \{1, 2, \ldots, n\} = [1, n]$, n is a positive integer. $A = (a_{ij})_{n\times n}$ is an $n \times n$ square matrix with $a_{ij} > 0$ for $(i, j) \in [1, n] \times [1, n]$.

A column vector $x = \mathrm{col}(x_1, x_2, \ldots, x_n) \in R^n$ is said to be a solution of (1.1) if substitution x into (1.1) renders it an identity. The vector x is said to be positive if $x_k > 0$ for $k \in [1, n]$, negative if $x_k < 0$ for $k \in [1, n]$, and non-zero if $x_k \neq 0$ for $k \in [1, n]$. Positive, negative and (strongly) non-zero vector x are denoted by $x > 0$, $x < 0$ and $x \neq 0$ respectively. Similarly, a matrix is said to be positive if all its entries are positive.

System (1.1) can be rewritten by a summability formula as the form

$$x_i = \lambda \sum_{j=1}^{n} a_{ij} f_j(x_j), \quad i \in [1, n], \tag{1.2}$$

which can be seen as the analogue of the Hammerstein integration equation

$$\psi(x) = \lambda \int_G K(x, y) f(\psi(y))\,\mathrm{d}y. \tag{1.3}$$

The importance of Eq. (1.3) is well known and it has been studied since 1930. However, to the best of our knowledge, little has been directly done for the nonlinear problems (1.1) or (1.2). In this paper, we consider the existence, uniqueness, multiplicity and nonexistence of positive solutions for (1.1) or (1.2).

Nonlinear systems of the form (1.1) or (1.2) arise in many applications. In Section 2, some problems in various areas are transformed into system (1.1) or (1.2). Then, in Section 3, we consider the existence and uniqueness of positive solutions. Furthermore, results on multiplicity and nonexistence are proved in Section 4.

2. Problems expressed by (1.1) or (1.2)

A large number of problems can be converted into system (1.1) or (1.2). In this section, we give some interesting facts on the transformation.

2.1. *Second order Dirichlet problems*

A second order difference equation of the form

$$\Delta^2 x_{k-1} + \lambda f_k(x_k) = 0, \quad k \in [1, n], \ \ \lambda > 0 \tag{2.1}$$

with the discrete boundary value condition

$$x_0 = 0 = x_{n+1} \tag{2.2}$$

has been extensively studied by a number of authors (see [1–4,13,24,29,51,52]). Indeed, problems (2.1) and (2.2) can be rewritten as a nonlinear system of the form

$$\begin{cases} 2x_1 - x_2 = \lambda f_1(x_1), \\ -x_1 + 2x_2 - x_3 = \lambda f_2(x_2), \\ \vdots \\ -x_{n-2} + 2x_{n-1} - x_n = \lambda f_{n-1}(x_{n-1}), \\ -x_{n-1} + 2x_n = \lambda f_n(x_n). \end{cases} \tag{2.3}$$

Denote $B = (b_{ij})$ for $i, j \in [1, n]$ with

$$b_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } j = i + 1 \text{ or } j = i - 1, \\ 0 & \text{otherwise,} \end{cases} \tag{2.4}$$

then system (2.3) can be written as $Bx = \lambda F(x)$. Let

$$g_{ij} = \begin{cases} \frac{j(n+1-i)}{n+1}, & 1 \leqslant j \leqslant i \leqslant n, \\ \frac{i(n+1-j)}{n+1}, & 1 \leqslant i \leqslant j \leqslant n, \end{cases}$$

the matrix B is invertible and $B^{-1} = A = (g_{ij})_{n\times n}$, which is a positive matrix.

2.2. *Third order difference equations*

Consider the third order difference equation

$$\Delta^3 x_{k-2} + \lambda f_k(x_k) = 0, \quad k \in [1, n], \quad \lambda > 0 \tag{2.5}$$

with the boundary value conditions

$$x_{-1} = x_0 = 0 = x_{n+1}. \tag{2.6}$$

The boundary value problems (2.5) and (2.6) were studied in [5,22]. Let $C = (c_{ij})$, $i, j \in [1, n]$ be the matrix defined by

$$c_{ij} = \begin{cases} 3 & \text{if } i = j, \\ -1 & \text{if } j = i + 1, \\ -3 & \text{if } j = i - 1, \\ 1 & \text{if } j = i - 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then problems (2.5) and (2.6) may be rewritten as $Cx = \lambda F(x)$. Note that the matrix C is invertible and $A = C^{-1} = (a_{ij})_{n\times n}$ is positive (see [22]).

2.3. *Fourth order difference equations*

Boundary value problems involving fourth order difference equations such as

$$\Delta^4 x_{k-2} - \lambda f_k(x_k) = 0, \quad k \in [1, n], \tag{2.7}$$

$$u_{-2} = u_{-1} = u_0 = 0 = u_{n+1} = u_{n+2} \tag{2.8}$$

were studied in [6,11,32]. Problems (2.7) and (2.8) can also be expressed by (1.1) or (1.2), where A is the inverse matrix of $D = (d_{ij})$, $i, j \in [1, n]$ and

$$d_{ij}=\begin{cases}6 & \text{if } i=j,\\ -4 & \text{if } j=i-1 \text{ or } j=i+1,\\ 1 & \text{if } j=i-2 \text{ or } j=i+2,\\ 0 & \text{otherwise.}\end{cases}$$

It is known that A is positive [6,11,32].

Boundary value problems for even order difference equations have been extensively studied (see [7,46,47]). Our theorems are also valid for such Dirichlet boundary value problems.

2.4. Three-point boundary value problems

Consider the three-point boundary value problem of the form

$$\begin{cases}\Delta^2 x_{k-1}+\lambda f_k(x_k)=0, & k\in[1,n],\\ x_0=0, & ax_l=x_{n+1},\end{cases} \tag{2.9}$$

where $n\in\{2,3,\ldots\}$, $l\in[1,n]$ and $f_k\in C(R^+,R^+)$. For Eq. (2.9), the existence of one or two positive solutions have been established in [49] by using a fixed point theorem. In fact, boundary value problem (2.9) can be expressed as

$$Ex=\lambda F(x), \tag{2.10}$$

where $x=\operatorname{col}(x_1,x_2,\ldots,x_n)$, $F(x)=\operatorname{col}(f_1(x_1),f_2(x_2),\ldots,f_n(x_n))$ and $E=(e_{ij})$, here

$$e_{ij}=\begin{cases}2 & \text{if } i=j,\\ -1 & \text{if } j=i-1 \text{ or } j=i+1,\\ -a & \text{if } i=n,\ \ j=l,\\ 0 & \text{otherwise.}\end{cases}$$

Boundary value problem (2.9) is equivalent to (2.10) in the sense that $\{0,x_1,x_2,\ldots,x_n,ax_l\}$ is a solution of (2.9) if, and only if, $\operatorname{col}(x_1,x_2,\ldots,x_n)$ is a solution of (2.10).

To apply fixed point theorems on the existence of solutions of (2.9) or (2.10), we naturally hope the matrix E is invertible. In the following, we prove the reversibility of E. In Section 3, some existence results of positive solutions of (2.9) or (2.10) are obtained by using fixed point theorems.

In fact, let F and H are $n\times n$ matrices such that $F=(f_{ij})$ and $H=(h_{ij})$, $i,j\in[1,n]$

$$f_{ij}=\begin{cases}2 & \text{if } i=j,\\ -1 & \text{if } j=i-1 \text{ or } j=i+1,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$h_{ij}=\begin{cases}-a & \text{if } i=n,\ \ j=l,\\ 0 & \text{otherwise,}\end{cases}$$

we have $E=F+H$ and

$$E^{-1}=(F+H)^{-1}=(F(I+F^{-1}H))^{-1}=(I+F^{-1}H)^{-1}F^{-1}.$$

Note that

$$
\begin{aligned}
F^{-1} &= \frac{1}{n+1}\begin{pmatrix}
n & n-1 & n-2 & \cdots & 2 & 1\\
n-1 & 2(n-1) & 2(n-2) & \cdots & 4 & 2\\
n-2 & 2(n-2) & 3(n-2) & \cdots & 6 & 3\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
2 & 4 & 6 & \cdots & 2(n-1) & n-1\\
1 & 2 & 3 & \cdots & n-1 & n
\end{pmatrix}\\
&= (g_{ij})_{n\times n},
\end{aligned}
$$

where

$$
g_{ij} = \begin{cases} \frac{j(n+1-i)}{n+1}, & 1 \leqslant j \leqslant i \leqslant n,\\ \frac{i(n+1-j)}{n+1}, & 1 \leqslant i \leqslant j \leqslant n. \end{cases} \tag{2.11}
$$

Let $R = (r_{ij})_{n\times n}$ be defined as

$$
r_{ij} = \begin{cases} i & \text{if } j = l,\\ 0 & \text{if } j \neq l. \end{cases}
$$

Thus,

$$
R = \begin{pmatrix}
0 & \cdots & 1 & \cdots & 0\\
0 & \cdots & 2 & \cdots & 0\\
 & \cdots & \cdots & \cdots & \\
0 & \cdots & (n-1) & \cdots & 0\\
0 & \cdots & n & \cdots & 0
\end{pmatrix}.
$$

Then, we have $F^{-1}H = \frac{-a}{n+1}R$. It is well known that

$$
(I + F^{-1}H)^{-1} = I + \sum_{k=1}^{\infty}(-1)^k(F^{-1}H)^k
$$

and

$$
(F^{-1}H)^k = l^{k-1}\left(\frac{-a}{n+1}\right)^k R.
$$

Furthermore, when $l|a| < n+1$,

$$
\begin{aligned}
\sum_{k=1}^{\infty}(-1)^k(F^{-1}H)^k &= \sum_{k=1}^{\infty} l^{k-1}\left(\frac{a}{n+1}\right)^k R\\
&= \frac{a}{n+1-l\alpha}R.
\end{aligned}
$$

Thus, we have

$$
(I + F^{-1}H)^{-1} = I + \frac{a}{n+1-la}R
$$

and

$$\begin{aligned}E^{-1}&=(I+F^{-1}H)^{-1}F^{-1}\\&=\left(I+\frac{a}{n+1-la}R\right)(g_{ij})_{n\times n}\\&=\left(g_{ij}+\frac{ia}{n+1-la}g_{lj}\right)_{n\times n}.\end{aligned}$$

Theorem 2.1. *If $|a| < (n+1)/l$, then the matrix E is invertible and its inversion is*

$$E^{-1}=\left(g_{ij}+\frac{ia}{n+1-la}g_{lj}\right)_{n\times n}. \tag{2.12}$$

In view of Theorem 2.1, system (2.9) can be rewritten as $x=\lambda E^{-1}F(x)$, or

$$x_i=\lambda\sum_{j=1}^{n}\left(g_{ij}+\frac{ia}{n+1-la}g_{lj}\right)f_j(x_j),\quad i=1,2,\ldots,n. \tag{2.13}$$

Naturally,

$$G(i,j)=g_{ij}+\frac{ia}{n+1-la}g_{lj},\quad 1\leqslant i,j\leqslant n \tag{2.14}$$

can be called the Green's function of problem (2.9), where $1\leqslant l\leqslant n$ is a fixed integer, a is a constant and the condition $|a|<(n+1)/l$ is satisfied. For $1\leqslant i,j\leqslant n$, g_{ij} is defined by (2.11).

In many existence problems, $G(i,j)>0$ for $1\leqslant i,j\leqslant n$ is usually asked. When $0\leqslant a<(n+1)/l$, it easily follows that $G(i,j)>0$ for $1\leqslant i,j\leqslant n$. In the following, we assume that $a<0$. Note that

$$\min_{1\leqslant i,j\leqslant n} g_{ij}=\frac{1}{n+1},$$

then we only need to consider the sign of

$$\frac{1}{n+1}+\frac{na}{n+1-la}g_{lj}. \tag{2.15}$$

By the definition of g_{lj}, we consider

$$\begin{aligned}\frac{1}{n+1}+\frac{na}{n+1-la}g_{lj}&=\begin{cases}\frac{1}{n+1}\left(1+\frac{naj(n+1-l)}{n+1-la}\right), & 1\leqslant j\leqslant l\leqslant n,\\ \frac{1}{n+1}\left(1+\frac{nal(n+1-j)}{n+1-la}\right), & 1\leqslant l\leqslant j\leqslant n,\end{cases}\\&\geqslant\frac{1}{n+1}\left(1+\frac{nal(n+1-l)}{n+1-la}\right)>0\end{aligned}$$

which implies that

$$a>-\frac{n+1}{l[n(n+1-l)-1]}. \tag{2.16}$$

Therefore, we have the following result:

Theorem 2.2. *Assume that*

$$-\frac{n+1}{l[n(n+1-l)-1]} < a < \frac{n+1}{l} \tag{2.17}$$

holds, then the Green's function $G(i, j)$ is positive for $1 \leqslant i, j \leqslant n$.

Thus, when the condition (2.17) holds, the three-point boundary value problem (2.9) can also be expressed by (1.1) or (1.2).

2.5. Dirichlet problem of partial difference equations

Boundary value problems involving partial difference equations arise from evaluating elliptic boundary value problems as well as vibrating nets, etc. ([12–14], Chapter 1 of [24,33]).

Following the terminologies of [24], let S be a (finite) net in the lattice plane and ∂S its exterior boundary. The discrete Laplacian D is defined by

$$Du(i, j) = u(i+1, j) + u(i-1, j) + u(i, j+1) + u(i, j-1) - 4u(i, j),$$

where $u(i, j)$ is a real function defined on $S \cup \partial S$. A common boundary value problem involving partial difference equations is of the form

$$\begin{cases} Du(w) + \lambda f_w(u(w)) = 0, & w \in S, \\ u(w) = 0, & w \in \partial S, \end{cases} \tag{2.18}$$

where λ is a positive parameter and $f_w \in C(R, R)$ for $w \in S$.

Cheng [13], Cheng-Lu [14] and Pao [36] considered the existence of positive solutions for (2.18) by eigenvalue, contraction and monotone methods. Continuous analogs have also been considered, see e.g. [8,16,17,20,30] and the listed references there.

Problem (2.18) can be expressed in the form (1.1) or (1.2) (see [13]). Roughly, let's denote the points in S by $z_1, z_2, \ldots, z_n$. Let $B = (b_{ij})$ be the adjacency matrix defined by $b_{ij} = 1$ if z_i and z_j have Euclidean distance 1 and $b_{ij} = 0$ otherwise. Then (2.18) can be written as

$$(A - 4I)u + \lambda G(u) = 0, \tag{2.19}$$

where I is the identity matrix, $u = \mathrm{col}(u(z_1), u(z_2), \ldots, u(z_n))$ and

$$G(u) = \mathrm{col}(f_{z_1}(u(z_1)), f_{z_2}(u(z_2)), \ldots, f_{z_n}(u(z_n))).$$

We may easily check [13] that the matrix $4I - A$ is positive definite and its inverse is positive.

2.6. Existence of periodic solutions

Recently, the existence of periodic positive solutions for the following equations have been studied (see [21,26,37,48,50,53,54]):

$$x_{n+1} = a_n x_n + \lambda f(x_n), \quad n \in Z, \tag{2.20}$$

where $\{a_n\}_{n \in Z}$ is a positive ω-periodic sequence satisfying $\prod_{s=0}^{\omega-1} a_s^{-1} > 1$, λ is a positive constant, and $f(u) : [0, \infty) \to [0, \infty)$ is a real continuous function.

For some positive integer p, we assume that Eq. (2.20) has a $p\omega$-periodic solution $\{x_n\}$. We proceed formerly from (2.20) and obtain

$$\Delta \left\{ x_n \prod_{k=-\infty}^{n-1} \frac{1}{a_k} \right\} = \lambda \prod_{k=-\infty}^{n} \frac{1}{a_k} f(x_n).$$

Summing the above formal equations from n to $n + p\omega - 1$, we obtain

$$x_n = \lambda \sum_{s=n}^{n+p\omega-1} G(n,s)f(x_s), \quad n \in Z, \tag{2.21}$$

where

$$G(n,s) = \left(\prod_{k=n}^{s} \frac{1}{a_k}\right)\left(\prod_{k=0}^{p\omega-1} \frac{1}{a_k} - 1\right)^{-1}, \quad n \in Z, \ \ n \leqslant s \leqslant n + p\omega - 1.$$

If $\{x_n\}$ is a $p\omega$-periodic solution of (2.21), then we have

$$\begin{aligned}
\frac{1}{a_n}x_{n+1} - x_n &= \lambda\left(\sum_{s=n+1}^{n+p\omega} \frac{1}{a_n}G(n+1,s)f(x_s) - \sum_{s=n}^{n+p\omega-1} G(n,s)f(x_s)\right) \\
&= \lambda\left(\left(\frac{1}{a_n}G(n+1,n+p\omega) - G(n,n)\right)f(x_n)\right) \\
&\quad \times \lambda\left(\sum_{s=n+1}^{n+p\omega-1}\left(\frac{1}{a_n}G(n+1,s) - G(n,s)\right)f(x_s)\right) \\
&= \lambda\left(\left(\frac{1}{a_n}G(n+1,n+p\omega) - G(n,n)\right)f(x_n)\right) \\
&= \lambda\frac{1}{a_n}f(x_n)
\end{aligned}$$

which implies

$$x_{n+1} = a_n x_n + \lambda f(x_n).$$

Note that we only concern with the existence of $p\omega$-periodic solution of (2.21). Thus, the existence of $p\omega$-periodic solution of (2.21) can also be expressed by system (1.1) or (1.2).

Similarly, in the case $\prod_{s=0}^{\omega-1} a_s^{-1} < 1$, we can consider the equation

$$x_{n+1} = a_n x_n - \lambda f(x_n). \tag{2.22}$$

2.7. Numerical solutions for differential equations

As a mathematical model for an adiabatic tubular chemical reactor which processes an irreversible exothermic chemical reaction, the following boundary value problems have been studied by a number of authors:

$$u'' - \lambda u' + \lambda\mu(\beta - u)\exp(u) = 0, \tag{2.23}$$

$$u'(0) = \lambda u(0), \quad u'(1) = 0. \tag{2.24}$$

In the above system, the unknown u represents the steady state temperature of the reaction, and the parameters λ, μ and β represents the Peclet number, the Damkohler number and the dimensionless adiabatic temperature rise, respectively [34]. Problems (2.23) and (2.24) can be numerically solved by applying the Newton's method. To do this, it is first changed to the following nonlinear system by using the SINC method [38]:

$$Ax = -\lambda\mu F(x), \tag{2.25}$$

where

$$x = \mathrm{col}(x_{-N-1}, x_{-N}, \dots, x_{N+1})$$

are the unknown coefficients,

$$F(x) = \mathrm{col}(f_{-N-1}(x_{-N-1}), f_{-N}(x_{-N}), \dots, f_{N+1}(x_{N+1}))$$

are the nonlinear functions defined by

$$f(x_i) = (\beta - x_j)\exp(x_j) \quad \text{for } j \in \{-N-1, -N, \dots, N+1\}.$$

Here, N is a positive integer. The $(2N+3) \times (2N+3)$ matrix $A = (a_{ij})$ contains the coefficients which can be found by the SINC method [38]. It is known that A is invertible. Thus, Eq. (2.25) can be changed to the form of system (1.1) or (1.2).

2.8. Steady state on a complex dynamical network

Recently, Li et al. [31] considered a complex network that consists of N identical linearly and diffusively coupled nodes, with each node being an m-dimensional dynamical system. The state equations of this dynamical network are given by

$$x_i' = f(x_i) + \sum_{j=1, j\neq i}^{N} c_{ij} a_{ij} \Gamma(x_j - x_i), \quad i = 1, 2, \dots, N, \tag{2.26}$$

where $x_i = (x_{i1}, x_{i2}, \dots, x_{im})^{\mathrm{T}} \in R^m$ are the state variables of node i, the constant $c_{ij} > 0$ represents the coupling strength between node i and node j, $\Gamma = (\tau_{ij}) \in R^{m\times m}$ is a matrix linking coupled variables, and if some pairs (i, j), $1 \leqslant i, j \leqslant m$, with $\tau_{ij} \neq 0$, then it means two coupled nodes are linked through their ith and jth state variables, respectively. In network (2.26), the coupling matrix $A = (a_{ij}) \in R^{N\times N}$ represents the coupling configuration of the network, which is assumed as a random network described by the E-R model (see [18,19]) or a scale-free network described by the B-A model (see [9,10,41,42,35]). If there is a connection between node i and node j $(i \neq j)$, then $a_{ij} = a_{ji} = 1$; otherwise, $a_{ij} = a_{ji} = 0$ $(i \neq j)$. If the degree k_i of node i is defined to be the number of its outreaching connections, then

$$\sum_{j=1, j\neq i}^{N} a_{ij} = \sum_{j=1, j\neq i}^{N} a_{ji} = k_i, \quad i = 1, 2, \dots, N.$$

Let the diagonal elements be $a_{ii} = -k_i$, $i = 1, 2, \dots, N$.

In [31], the authors assumed there exists a generous stationary state for network (2.26) which is defined as

$$x_1 = x_2 = \cdots = x_N = \overline{x}, \quad f(\overline{x}), \tag{2.27}$$

and they apply the pinning control strategy on a small fraction of the nodes to achieve the stabilization control of the goal (2.27). We can rewrite the network (2.26) by the system

$$X' = -DX + F(X), \tag{2.28}$$

where $X = (x_1, x_2, \ldots, x_{mN})^{\mathrm{T}}$ is the state vector, the $F(X)$ denotes the mN-dimensional functional value vector of X, and

$$D = (d_{ij})_{mN \times mN}$$

is an $mN \times mN$ matrix, in [31] it is nonnegative definite and satisfied the conditions

$$d_{ij} = d_{ji} \leqslant 0, \quad i \neq j, \;\; i, j = 1, 2, \ldots, mN. \tag{2.29}$$

In (2.26), if the sub-systems are described by the Hopfield's networks

$$x_i' = -\alpha_i x_i + f(x_i), \quad i = 1, 2, \ldots, m,$$

we have

$$x_i' = -\alpha_i x_i + f(x_i) + \sum_{j=1, j \neq i}^{N} c_{ij} a_{ij} \Gamma(x_j - x_i), \quad i = 1, 2, \ldots, N,$$

then the matrix D is clearly positive definite. In this case, we have known that the steady state on a complex Hopfield's networks can be expressed by

$$(\boldsymbol{\alpha} + A)X = \lambda F(X), \tag{2.30}$$

where $\boldsymbol{\alpha} = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n) > 0$, $A = (a_{ij})$ satisfies the conditions

$$a_{ii} > 0 \;\text{ for } i \in [1, n], \quad a_{ij} \leqslant 0 \;\text{ for } i, j \in [1, n] \text{ and } i \neq j$$

and

$$-a_{ii} = \sum_{j=1, j \neq i}^{n} a_{ij} = \sum_{j=1, j \neq i}^{n} a_{ji}.$$

Clearly, $\boldsymbol{\alpha} + A$ is a positive definite M-matrix. On the other hand, we only concern the coupled systems. Thus, we can get that the matrix $(\boldsymbol{\alpha} + A)^{-1}$ is positive [23]. System (2.30) then can be written by (1.1) or (1.2). On the other hand, the existence of stationary state solutions for discrete dynamical systems is also very important, see Shi and Chen [39,40] and the listed references. Thus, our existence results are very usefulness for the complex study of discrete dynamical systems or discrete space and continuous time dynamical systems.

3. Existence and uniqueness

For motivating to consider our main results, we firstly see some simple facts. When $n = 1$, problem (1.1) is reduced to

$$x = \lambda a f(x), \tag{3.1}$$

where $a > 0$. Particularly, we let $f(x) = x^\alpha$, it is well known that Eq. (3.1) has a unique positive solution for any positive number λ when $\alpha > 1$ or $0 < \alpha < 1$. Naturally, we hope to extend the above result to the general case.

In this section, we will assume that $F(x) = x^\alpha = \mathrm{col}(x_1^\alpha, x_2^\alpha, \ldots, x_n^\alpha)$, where $0 < \alpha < 1$ or $\alpha > 1$. That is, we will consider the uniqueness and existence of positive solutions for the nonlinear algebraic system of the form

$$x = \lambda A x^\alpha. \tag{3.2}$$

Theorem 3.1. *For any* $\lambda > 0$ *and* $0 < \alpha < 1$, *system* (3.2) *has an unique positive solution.*

Proof. For any $x \in R^n$, define $\|x\| = \max_{i\in[1,n]} |x_i|$, then R^n is a Banach space. Let

$$S = \left\{x \in R^n | (\lambda a_{ii})^{\frac{1}{1-\alpha}} \leqslant x_i \leqslant (\lambda\|A\|)^{\frac{1}{1-\alpha}}, 1 \leqslant i \leqslant n\right\}$$

(where $\|A\| = \max_{i,j\in[1,n]} a_{ij}$) and set $Tx = \lambda Ax^\alpha$ for any $x \in S$. Then S is a bounded closed set and $T(S) \subset S$. In fact, if $x \in S$, then

$$\|Tx\| = \|\lambda Ax^\alpha\| \leqslant \lambda\|A\|\|x\|^\alpha \leqslant \lambda\|A\|(\lambda\|A\|)^{\frac{\alpha}{1-\alpha}} = (\lambda\|A\|)^{\frac{1}{1-\alpha}}$$

and

$$(Tx)_i = \lambda \sum_{j=1}^{n} a_{ij}x_j^\alpha \geqslant \lambda a_{ii}x_i^\alpha \geqslant \lambda a_{ii}(\lambda a_{ii})^{\frac{\alpha}{1-\alpha}} = (\lambda a_{ii})^{\frac{1}{1-\alpha}}.$$

Since the mapping $T : S \to S$ is continuous, T has a fixed point in S by Brouwer's theorem.

Let x and y be two positive solutions and

$$\rho = \min_{i\in[1,n]} \frac{x_i}{y_i} = \frac{x_{i_0}}{y_{i_0}}.$$

Then $\rho > 0$ and

$$\rho = \frac{\lambda\sum_{j=1}^n a_{i_0j}x_j^\alpha}{\lambda\sum_{j=1}^n a_{i_0j}y_j^\alpha} \geqslant \min_{i\in[1,n]} \frac{x_i^\alpha}{y_i^\alpha} = \rho^\alpha.$$

Hence $\rho^{1-\alpha} \geqslant 1$ or $\rho \geqslant 1$, which implies $x \geqslant y$. Interchanging the role x and y, we have $y \geqslant x$. We thus obtain $x = y$. Therefore, the proof is completed. □

Remark 3.2. When the matrix A is reduced to the case of Section 2.1, Theorem 3.1 is the main result in [15]. However, it is new for (2.5)–(2.9), (2.18), (2.20), (2.22) and (2.30).

Remark 3.3. Suppose that α and A are given. If w is the unique solution of $x = Ax^\alpha$, then the solution of (3.2) can be expressed as $u_\lambda = \lambda^{1/(1-\alpha)}w$. Thus, the positive solutions of (3.2) are continuous with respect to λ, i.e., $\lambda \to \lambda_0 > 0$ implies that $\|u_\lambda - u_{\lambda_0}\| \to 0$, where $\|\cdot\|$ denotes the norm of vectors. In addition, it can be obtained that $\lim_{\lambda\to 0} u_\lambda = 0$ and $\lim_{\lambda\to\infty} u_\lambda = \infty$.

Remark 3.4. When $\alpha > 1$, existence of positive solutions can be proved but uniqueness is not a fact. An example is given below.

Example 3.5. Consider the problem

$$\begin{cases} \Delta_x^2 u_{i-1,j} + \Delta_y^2 u_{i,j-1} + \lambda u_{i,j}^3 = 0 & (i,j) \in \overline{S}, \\ u_{i,j} = 0 & (i,j) \in \partial\overline{S} \end{cases} \tag{3.3}$$

which can be rewritten by the system of the form

$$\begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{1,2} \\ u_{2,1} \\ u_{2,2} \end{pmatrix} = \lambda \begin{pmatrix} u_{1,1}^3 \\ u_{1,2}^3 \\ u_{2,1}^3 \\ u_{2,2}^3 \end{pmatrix} \tag{3.4}$$

or

$$\begin{pmatrix} u_{1,1} \\ u_{1,2} \\ u_{2,1} \\ u_{2,2} \end{pmatrix} = \lambda \begin{pmatrix} \frac{7}{24} & \frac{1}{12} & \frac{1}{12} & \frac{1}{24} \\ \frac{1}{12} & \frac{7}{24} & \frac{1}{24} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{24} & \frac{7}{24} & \frac{1}{12} \\ \frac{1}{24} & \frac{1}{12} & \frac{1}{12} & \frac{7}{24} \end{pmatrix} \begin{pmatrix} u_{1,1}^3 \\ u_{1,2}^3 \\ u_{2,1}^3 \\ u_{2,2}^3 \end{pmatrix}. \tag{3.5}$$

When $\lambda = 1$, the following numerical positive solutions of (3.3), (3.4) or (3.5) can be found by using Maple:

$$\begin{aligned}
&u_{11} = 0.58245, \quad u_{22} = 0.58245, \quad u_{12} = 1.8344, \quad u_{21} = 0.29783;\\
&u_{11} = 0.58245, \quad u_{22} = 0.58245, \quad u_{12} = 0.29783, \quad u_{21} = 1.8344;\\
&u_{11} = 0.29783, \quad u_{22} = 1.8344, \quad u_{12} = 0.58245, \quad u_{21} = 0.58245;\\
&u_{11} = 1.8344, \quad u_{22} = 0.29783, \quad u_{12} = 0.58245, \quad u_{21} = 0.58245;\\
&u_{11} = 1.4142, \quad u_{22} = 1.4142, \quad u_{12} = 1.4142, \quad u_{21} = 1.4142;\\
&u_{11} = 0.61803, \quad u_{22} = 1.618, \quad u_{12} = 0.61803, \quad u_{21} = 1.618;\\
&u_{11} = 0.61803, \quad u_{22} = 1.618, \quad u_{12} = 1.618, \quad u_{21} = 0.61803;\\
&u_{11} = 1.618, \quad u_{22} = 0.61803, \quad u_{12} = 0.61803, \quad u_{21} = 1.618;
\end{aligned}$$

and

$$u_{11} = 1.618, \quad u_{22} = 0.61803, \quad u_{12} = 1.618, \quad u_{21} = 0.61803.$$

4. Existence, multiplicity and nonexistence

In this section, we shall show that the number of positive solutions of (1.1) or (1.2) can be determined by the asymptotic behaviors of the quotient of $f_i(u)/u$ at zero and infinity. Our arguments are based on a well-known fixed point theorem. Similar arguments have employed in [20,43–45] to prove analogous results for existence, multiplicity and nonexistence of positive solutions of boundary-value problems or periodic problems. However, some of our results are different from the corresponding boundary-value problems of ordinary or partial differential equations (see, Theorems 5–8 and 10). In the following, we will assume that $f_i : [0, \infty) \to [0, \infty)$ are continuous for $i \in [1, n]$.

4.1. Preliminaries

For any $x \in R^n$, define $\|x\| = \max_{i\in[1,n]} |x_i|$, then R^n is a Banach space. Let

$$m = \min_{i,j\in[1,n]} a_{ij}, \quad M = \max_{i,j\in[1,n]} a_{ij} \quad \text{and} \quad \sigma = \frac{m}{M}.$$

Define P and K are two cones of R^n respectively

$$P = \{x \in R^n : x_i \geqslant 0, i \in [1, n]\}, \quad K = \{x \in R^n : x_i \geqslant \sigma\|x\|, i \in [1, n]\}.$$

Also, for a positive number r, define Ω_r by

$$\Omega_r = \{x \in K : \|x\| < r\}.$$

Note that $\partial\Omega_r = \{x \in K : \|x\| = r\}$. Let $T_\lambda : P \to R^n$ be the map defined by

$$(T_\lambda x)_i = \lambda \sum_{j=1}^{n} a_{ij} f_j(x_j), \quad i \in [1, n] \tag{4.1}$$

or

$$T_\lambda x = \lambda A F(x). \tag{4.2}$$

Existence of solutions of (1.1) or (1.2) is equivalent to the fixed point problem of T_λ. Some preliminaries are given below:

(i) For $x \in P$, we have

$$(T_\lambda x)_i \geqslant m\lambda \sum_{j=1}^{n} f_j(x_j) \geqslant \sigma \|x\|, \quad i \in [1, n]$$

which implies that $T_\lambda P \subset K$.

(ii) If for $x \in K$, there exists $i_0 \in [1, n]$ and $\eta_{i_0} > 0$ such that $f_{i_0}(x_{i_0}) \geqslant \eta_{i_0} x_{i_0}$, then we have

$$(T_\lambda x)_i \geqslant m\lambda \eta_{i_0} x_{i_0} \geqslant \lambda \sigma m \eta_{i_0} \|x\|.$$

Thus, $\|T_\lambda x\| \geqslant \lambda \sigma m \eta_{i_0} \|x\|$.

(iii) If for $r > 0$, $x \in \partial\Omega_r$, there exists δ_i such that $f_i(x_i) \leqslant \delta_i x_i$, then

$$\|T_\lambda x\| \leqslant M\lambda \sum_{j=1}^{n} \delta_j x_j \leqslant \lambda M \sum_{j=1}^{n} \delta_j \|x\|.$$

(iv) For $r > 0$, we define

$$Q_i(r) = \max_{0 \leqslant t \leqslant r} f_i(t), \quad q_i(r) = \min_{\sigma r \leqslant t \leqslant r} f_i(t).$$

If $x \in \partial\Omega_r$, then

$$\|T_\lambda x\| \geqslant \lambda m \sum_{j=1}^{n} f_j(x_j) \geqslant \lambda m \sum_{j=1}^{n} q_j(r) = \lambda m q(r)$$

and

$$\|T_\lambda x\| \leqslant \lambda M \sum_{j=1}^{n} f_j(x_j) \leqslant \lambda M \sum_{j=1}^{n} Q_j(r) = \lambda M Q(r).$$

(v) Let E be a Banach space and K be a cone of E. For $r > 0$, define $K_r = \{u \in K : \|u\| < r\}$. Assume that $T : \overline{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_k = \{u \in K : \|u\| = r\}$. If $\|Tx\| \geqslant \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 0$. If $\|Tx\| \leqslant x$ for $x \in \partial K_r$, then $i(T, K_r, K) = 1$. The proof of this theorem can be seen in [27].

4.2. *Existence and multiplicity*

We make use of the following notations:

$$f_i^{(0)} = \lim_{u \to 0^+} \frac{f_i(u)}{u} \quad \text{and} \quad f_i^{(\infty)} = \lim_{u \to \infty} \frac{f_i(u)}{u} \quad \text{for } i \in [1, n].$$

Theorem 4.1. *For any $i \in [1, n]$, if $f_i^{(0)} = 0$ or $f_i^{(\infty)} = 0$, then there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, (1.1) or (1.2) has a positive solution. Moreover, if $f_i^{(0)} = f_i^{(\infty)} = 0$ for $i \in [1, n]$, there exists $\lambda_0 > 0$, such that for all $\lambda > \lambda_0$, (1.1) or (1.2) has two positive solutions.*

Proof. Let $r_1 = 1$ and $\lambda_0 = 1/\lambda m q(r_1)$, in view of (iv) we have

$$\|T_\lambda x\| > \|x\| \quad \text{for } x \in \partial\Omega_{r_1} \text{ and } \lambda > \lambda_0.$$

If for any $i \in [1, n]$, $f_i^{(0)} = 0$, we can choose $0 < r_2 < r_1$ so that $f_i(x_i) \leqslant \delta_i x_i$ for $0 \leqslant x_i \leqslant r_2$, where the constants δ_i satisfy

$$\lambda M \sum_{j=1}^{n} \delta_j < 1.$$

By (iii), we obtain that

$$\|T_\lambda x\| \leqslant \lambda M \sum_{j=1}^{n} f_j(x_j) \leqslant \lambda M \sum_{j=1}^{n} \delta_j x_j < \|x\| \quad \text{for } x \in \partial\Omega_{r_2}.$$

It follows from (v) that

$$i(T_\lambda, \Omega_{r_1}, K) = 0 \quad \text{and} \quad i(T_\lambda, \Omega_{r_2}, K) = 1.$$

Thus $i(T_\lambda, \Omega_{r_1} \backslash \overline{\Omega}_{r_2}, K) = -1$ and T_λ has a fixed point in $\Omega_{r_1} \backslash \overline{\Omega}_{r_2}$, which is a positive solution of (1.1) or (1.2) for $\lambda > \lambda_0$.

If for any $i \in [1, n]$, $f_i^{(\infty)} = 0$, there are $H > 0$ and $\delta_i > 0$ such that $f_i(x_i) \leqslant \delta_i x_i$ for $x_i \geqslant H$, where the constants satisfy $\lambda M \sum_{j=1}^{n} \delta_j < 1$. Let

$$r_3 = \max\left\{2r_1, \frac{H}{\sigma}\right\}$$

and it follows that $x_i \geqslant \sigma\|x\| \geqslant H$,

$$\|T_\lambda x\| \leqslant \lambda M \sum_{j=1}^{n} f_j(x_j) \leqslant \lambda M \sum_{j=1}^{n} \delta_j x_j < \|x\| \quad \text{for } x \in \partial\Omega_{r_3}.$$

Again, it follows from (v) that

$$i(T_\lambda, \Omega_{r_1}, K) = 0 \quad \text{and} \quad i(T_\lambda, \Omega_{r_3}, K) = 1.$$

Therefore, $i(T_\lambda, \Omega_{r_1} \backslash \overline{\Omega}_{r_3}, K) = 1$ and T_λ has a fixed point in $\Omega_{r_1} \backslash \overline{\Omega}_{r_3}$, which is a positive solution of (1.1) or (1.2) for $\lambda > \lambda_0$.

If $f_i^{(0)} = f_i^{(\infty)} = 0$ for $i \in [1, n]$, from the above proof, it is easy to see that T_λ has a fixed point $x^{(1)}$ in $\Omega_{r_1} \backslash \overline{\Omega}_{r_2}$ and a fixed point $x^{(2)}$ in $\Omega_{r_1} \backslash \overline{\Omega}_{r_3}$ such that

$$r_2 < \|x^{(1)}\| < r_1 < \|x^{(2)}\| < r_3.$$

Consequently, (1.1) or (1.2) has two positive solutions for $\lambda > \lambda_0$. □

Theorem 4.2. *If there exists $i_0 \in [1, n]$ such that $f_{i_0}^{(0)} = \infty$ or $f_{i_0}^{(\infty)} = \infty$, then there exists λ_0 such that for all $0 < \lambda < \lambda_0$, (1.1) or (1.2) has a positive solution. If there exist $i_0, j_0 \in [1, n]$ such that $f_{i_0}^{(0)} = f_{j_0}^{(\infty)} = \infty$, then there exists λ_0 such that for all $0 < \lambda < \lambda_0$, (1.1) or (1.2) has two positive solutions.*

Proof. Let $r_1 = 1$ and $\lambda_0 = 1/\lambda m Q(r_1)$, in view of (iv) we have

$$\|T_\lambda x\| < \|x\| \quad \text{for } x \in \partial\Omega_{r_1} \text{ and } 0 < \lambda < \lambda_0.$$

If $f_{i_0}^{(0)} = \infty$, there is a positive number $r_2 < r_1$ such that $f_{i_0}(x_{i_0}) \geqslant \eta_{i_0} x_{i_0}$ for $0 \leqslant x_{i_0} \leqslant r_2$, where $\eta_{i_0} > 0$ is chosen such that $\lambda m \sigma \eta_{i_0} > 1$.

$$\|T_\lambda x\| \geqslant \lambda m f_{i_0}(x_{i_0}) \geqslant \lambda m \eta_{i_0} x_{i_0} \geqslant \lambda m \sigma \eta_{i_0} \|x\| > \|x\| \quad \text{for } x \in \partial\Omega_{r_2}.$$

It follows from (v) that

$$i(T_\lambda, \Omega_{r_1}, K) = 1 \quad \text{and} \quad i(T_\lambda, \Omega_{r_2}, K) = 0.$$

Thus $i(T_\lambda, \Omega_{r_1} \backslash \overline{\Omega}_{r_2}, K) = 1$ and T_λ has a fixed point in $\Omega_{r_1} \backslash \overline{\Omega}_{r_2}$, which is a positive solution of (1.1) or (1.2) for $0 < \lambda < \lambda_0$.

If $f_{i_0}^{(\infty)} = \infty$, there is $H > 0$ such that $f_{i_0}(x_{i_0}) \geqslant \eta_{i_0} x_{i_0}$ for $x_{i_0} \geqslant H$, where η_{i_0} is chosen such that $\lambda m \sigma \eta_{i_0} > 1$. Let

$$r_3 = \max\left\{2r_1, \frac{H}{\sigma}\right\},$$

then

$$\|T_\lambda x\| \geqslant \lambda m \sigma \eta_{i_0} \|x\| > \|x\| \quad \text{for } x \in \partial\Omega_{r_3}.$$

It follows from (v) that

$$i(T_\lambda, \Omega_{r_1}, K) = 1 \quad \text{and} \quad i(T_\lambda, \Omega_{r_2}, K) = 0.$$

Thus $i(T_\lambda, \Omega_{r_1} \backslash \overline{\Omega}_{r_3}, K) = -1$ and T_λ has a fixed point in $\Omega_{r_1} \backslash \overline{\Omega}_{r_3}$, which is a positive solution of (1.1) or (1.2) for $0 < \lambda < \lambda_0$.

If $f_{i_0}^{(0)} = f_{j_0}^{(\infty)} = \infty$, it is easy to see from the above proof that T_λ has a fixed point $x^{(1)}$ in $\Omega_{r_1} \backslash \overline{\Omega}_{r_2}$ and a fixed point $x^{(2)}$ in $\Omega_{r_1} \backslash \overline{\Omega}_{r_3}$ such that

$$r_2 < \|x^{(1)}\| < r_1 < \|x^{(2)}\| < r_3.$$

Consequently, (1.1) or (1.2) has two positive solutions for $0 < \lambda < \lambda_0$ if there exist $i_0, j_0 \in [1, n]$ such that $f_{i_0}^{(0)} = f_{j_0}^{(\infty)} = \infty$. $\square$

The following results can be obtained similarly.

Theorem 4.3. *If $f_i^{(0)} = 0$ for $i \in [1, n]$ and there exists $i_0 \in [1, n]$ such that $f_{i_0}^{(\infty)} = \infty$ or there exists $i_0 \in [1, n]$ such that $f_{i_0}^{(0)} = \infty$ and $f_i^{(\infty)} = 0$ for $i \in [1, n]$. Then (1.1) or (1.2) has a positive solution for all $\lambda > 0$.*

Theorem 4.4. *Assume that $f_i^{(0)} \neq 0$ and $f_i^{(\infty)} \neq 0$ for $i \in [1, n]$. If there exist $i_0, j_0 \in [1, n]$ such that $f_{i_0}^{(0)} < \infty$ and $f_{j_0}^{(\infty)} < \infty$, then there exist two positive numbers $\lambda_1 < \lambda_2$, such that (1.1) or (1.2) has a positive solution for all $\lambda \in (\lambda_1, \lambda_2)$.*

Remark 4.5. Theorems 4.1–4.4 extend and improve some known results for the particular cases of (1.1) or (1.2). For references, please see [1–4,13,24,29,51,52] on boundary value problems of second order difference equations; [49] on three-point boundary value problems; [6,7,11,32,46,47] on fourth and even order difference equations; [12–14] on Dirichlet problem of partial difference equations; [21,48,50] on existence of positive periodic solutions.

4.3. Nonexistence

In this subsection, some nonexistence results are obtained.

Theorem 4.6. *Assume that there exist* $i_0 \in [1, n]$ *and* $j_0 \in [1, n]$ *such that* $f_{i_0}^{(0)} > 0$ *and* $f_{j_0}^{(\infty)} > 0$. *Then there exists* $\lambda_0 > 0$ *such that* (1.1) *or* (1.2) *has no positive solution for all* $\lambda > \lambda_0$.

Proof. Assume that there exist $i_0 \in [1, n]$ and $j_0 \in [1, n]$ such that $f_{i_0}^{(0)} > 0$ and $f_{j_0}^{(\infty)} > 0$. It follows that there exist positive numbers $\eta_{i_0}, \eta_{j_0}, r_1$ and r_2, such that $r_1 < r_2$ and

$$f_{i_0}(x_{i_0}) \geqslant \eta_{i_0} x_{i_0} \quad \text{for } x_{i_0} \in [0, r_1],$$
$$f_{j_0}(x_{j_0}) \geqslant \eta_{j_0} x_{j_0} \quad \text{for } x_{j_0} \geqslant r_2.$$

Let

$$c = \min\left\{\eta_{i_0}, \eta_{j_0}, \min_{r_1 \leqslant u \leqslant r_2} \frac{f_{j_0}(u)}{u}\right\},$$

then

$$f(u) = \begin{cases} f_{i_0}(u), & 0 \leqslant u \leqslant r_1 \\ f_{j_0}(u), & u > r_1 \end{cases}$$

which implies that $f(u) \geqslant cu$ for $u \in [0, \infty)$. Now, we assume that x is a positive solution of (1.1) or (1.2). Then

$$\|x\| = \|T_\lambda x\| \geqslant \lambda mc\sigma \|x\| > \|x\| \quad \text{for } \lambda > \frac{1}{mc\sigma},$$

which is a contradiction if $\lambda > \lambda_0 = 1/(mc\sigma)$. □

Theorem 4.7. *If for all* $i \in [1, n]$, $f_i^{(0)} < \infty$ *and* $f_i^{(\infty)} < \infty$, *then there exists* $\lambda_0 > 0$ *such that* (1.1) *or* (1.2) *has no positive solution for all* $0 < \lambda < \lambda_0$.

Proof. If $f_i^{(0)} < \infty$ and $f_i^{(\infty)} < \infty$. It follows that there exist positive numbers δ_i, r_1 and r_2, such that $r_1 < r_2$ and

$$f_i(x_i) \leqslant \delta_i x_i \quad \text{for } x_i \in [0, r_1],$$
$$f_i(x_i) \leqslant \delta_i x_i \quad \text{for } x_i \geqslant r_2.$$

Let

$$\bar{c} = \max\left\{\delta_i, \max_{r_1 \leqslant u \leqslant r_2} \frac{f_i(u)}{u}\right\},$$

then

$$f_i(u) \leqslant \bar{c}u \quad \text{for } u \in [0, \infty).$$

Assume x is a positive solution of (1.1) or (1.2), then

$$\|x\| = \|T_\lambda x\| \leqslant \lambda M\bar{c}\|x\| < \|x\| \quad \text{for } 0 < \lambda < \frac{1}{M\bar{c}},$$

which is a contradiction if $0 < \lambda < \lambda_0 = 1/(M\bar{c})$. □

The following result can be obtained similarly.

Theorem 4.8. *If $f_i^{(0)} \neq 0$ and $f_i^{(\infty)} \neq 0$ for $i \in [1, n]$ and there exist $i_0, j_0 \in [1, n]$ such that $f_{i_0}^{(0)} < \infty$ and $f_{j_0}^{(\infty)} < \infty$. Then there exist positive numbers $\lambda_1 < \lambda_2$ such that (1.1) or (1.2) has no positive solution for $\lambda < \lambda_1$ or $\lambda > \lambda_2$.*

Remark 4.9. Theorems 4.6–4.8 are also new for the particular cases of (1.1) or (1.2).

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